5. Consider the system of two equations:
$$\begin{aligned} x_1 + 2x_2 - x_3 + 4x_4 &= 1 \\ -x_1 - 2x_2 + 4x_3 - x_4 &= 5 \end{aligned}$$

(a) Solve this problem by setting it up as a system of 2 equations in 4 unknowns. Write the augmented matrix which results from this process. Put your answer in the box.

(b) Find the solution vector $x = (x_1, x_2, x_3, x_4)^T$. Express this as a column vector in the box below.

(c) This solution can be expressed as $x = x_h + x_p$ where x_h is the general solution to the homogeneous equation and x_p is a solution to the original equation. Find these.

(d) Describe the geometry of the solution set.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 1 \\ -1 & -2 & 4 & -1 & 5 \end{array}\right] \quad R_2 \to R_2 + R_1$$

$$R_2 \to \tfrac{1}{3} R_2 \left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 1 \\ 0 & 0 & 3 & 3 & 6 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

let $\boxed{x_4 = t}$

$x_3 + x_4 = 2$

$\boxed{x_3 = 2 - t}$

$x_1 + 2x_2 - x_3 + 4x_4 = 1$

let $x_2 = s$

$x_1 + 2s - (2-t) + 4t = 1$

$x_1 = 3 - 5t - 2s$

$$x = \begin{pmatrix} 3-5t-2s \\ s \\ 2-t \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5t-2s \\ s \\ -t \\ t \end{pmatrix}$$

$X_p + X_h$

2 free parameters $\Rightarrow$ solution is a plane,

6. Given the linear system
$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= b \\ -x_2 - x_3 &= a \\ x_1 + 4x_2 + 6x_3 &= 0 \end{aligned}$$

(a) ~~Solve this problem by~~ Set~~ting~~ it up as a system of 3 equations in 3 unknowns (x_1, x_2, x_3).

(b) What is the rank of the coefficient matrix?

(c) What is the nullity of the coefficient matrix?

(d) Write down the condition on a and b which guarantees a solution exists.

(e) If this condition is satisfied, will the solution be unique. Why or why not?

(a)
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & b \\ 0 & -1 & -1 & a \\ 1 & 4 & 6 & 0 \end{array}\right] R_3 \to R_3 - R_1$$
A

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & b \\ 0 & -1 & -1 & a \\ 0 & 2 & 2 & -b \end{array}\right] R_3 \to R_3 + 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & b \\ 0 & -1 & -1 & a \\ 0 & 0 & 0 & 2a-b \end{array}\right]$$

pivots = 2

(b) Rank(A) = 2

(c) nullity(A) = # columns − Rank = 1

(d) last equation implies $\boxed{2a - b = 0}$

(e) The solution will not be unique because rank(A) < 3 or (nullity(A) > 0).