Sample Questions For Calculus I Final (Fall 2001)

Wednesday, December 19, 6:00 - 8:00 PM, North Underground Lecture Hall

- **Skills**
  - **Limits**

    1. (a) $\lim_{x\to 0}\frac{1-e^{2x}}{x^2}$ (b) $\lim_{x\to 0}\frac{\sin(5x)}{x}$ (c) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$
    2. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{7n^3}$ given that $\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$.
    3. $\lim_{x\to 0}(1+2x)^{\frac{1}{x}}$ **Hint:** Take the natural logarithm of this limit.

  - **Diffferentiation**

    1. Determine the derivative of
       (a) $y=e^{\frac{1}{x}}$ (b) $y=x\,e^{2x}$ (b) $y=3^x$
    2. Find $\frac{dy}{dx}$ if $y=\frac{4x+1}{x^2-5}$.
    3. Evaluate the following:
       (a) $\frac{d}{dx}\,(\cos(3x)+x\,\sin(x))$ (b) $\frac{d}{dt}\,\sin^{-1}(3t)$ (c) $\frac{d}{dx}\,\ln(x+x^2)$
    4. Let $f(x)=\sin x$ and $g(x)=\sqrt{x}$.
       (a) Determine $f(g(x))$.
       (b) Determine the derivative of $f(g(x))$.
    5. The point (3,1) is on the graph of the curve whose points satisfy the equation $2xy-y^2=5$
       (a) Find and expression to compute $\frac{dy}{dx}$ at points on that curve where the tangent line is not vertical. **Hint** Implicit Differentiation
       (b) Write an equation for the line tangent to this curve at the point (3,1).

  - **Integration**

    1. Evaluate the following definite integrals
       (a) $\int_0^2 |2x-3|\,dx$ (b) $\int_0^{\pi/4}\sec^2\theta\,d\theta$
    2. Determine by substitution and state the specific substitution you are using.
       (a) $\int x\sqrt{3-x^2}\,dx$ (b) $\int x\cos(3x^2)\,dx$
       (c) $\int \frac{2x}{x^2+1}\,dx$ (d) $\int e^{\cos x}\sin x\,dx$
    3. Find the differentiable function g that has the properties:

$$g'(x)=x-6\cos(2x)\quad\text{for all } x\quad\text{and}\quad g(\pi)=1.$$

- **Related Rates**

  1. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/s. How rapidly is the area enclosed by the ripple increasing when the area is 30 square feet.
  2. A conical water tank with vertex down has a radius of 10 feet at the top and is 24 feet high. If water is being drained from the tank at a constant rate of 20 ft^3/min, how fast is the depth of the water changing when the water is 16 feet deep.

- **Local Linear Approximation** Use an appropriate local linear approximation to estimate the value of the given quantity.

  (a) $\sqrt{24}$ (b) $\sin(0.1)$

- **Analysis of functions** Let f be the polynomial function defined by $f(x) = x^4 - \frac{3}{8}x^2$ for all x.

  1. $f'(x) =$
  2. $f''(x) =$
  3. Find all critical points of f and classify the behavior of f at each point.
  4. Locate all inflection points of f.
  5. On which interval(s) is f decreasing? Describe the relevant evidence.
  6. On which interval(s) is the graph of f concave down? Describe the relevant evidence.
  7. Write an equation for a line tangent to the graph of f at an inflection point.
  8. Sketch the graph of f over an appropriate interval - indicate scales (or viewing window) for your plot; include graph of the tangent line you found in part (g).

- **Rectilinear Motion**

  1. The position function of a particle moving along a coordinate line is given by $s(t) = 2t^3 - 21t^2 + 60t + 3$, $t \geq 0$, where s is in feet and t is in seconds.
     (a) Find the velocity and acceleration functions.
     (b) At what time is the particle stopped?
     (c) When is the particle speeding up? Slowing down?
     (d) Find the total distance traveled by the particle from time $t = 0$ to $t = 5$.
  2. A ball is thrown from the surface of the earth (where the acceleration of gravity is 32 ft/sec^2) straight up into the air with an initial velocity of 60 mph (= 88 ft/sec). How high does the ball go?

- **Max/Min**

  1. Consider the function $f(x) = \dfrac{1}{x(1-x)}$ over the open interval (0,1).
     (a) Use a limit argument to show that this function has no absolute maximum on this interval.
     (b) What is the absolute minimum of this function on this interval and where does it occur. Justify your answer.
  2. A cylindrical can, open at the top, is to hold 500 cm^3 of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can. Be sure to state the domain of the function you are minimizing and show that the resulting value does indeed result in a minimum.
  3. There are many rectangles whose area is 100 square feet: for instance, one such rectangle is four feet wide and twenty-five feet tall, another is two hundred feet wide and one-half foot tall. Use ideas of calculus to show that in the set of all rectangles having area 100 square feet, the ten-by-ten square has the shortest perimeter.

- **Definitions**

  - **The Derivative**

    1. Limit Definition of Derivative
       (a) State a formal (limit) definition of the derivative of a function f.

(b) Find the derivative of the function $f(x) = 3x^2 - 2x$ using the definition in part (a). Show all of the steps in the limiting process.

2. The following limit represents $f'(a)$ for some function f and some number a. Find f and a.

$$\lim_{\Delta x \to 0} \frac{\sqrt{1+\Delta x} - 1}{\Delta x}$$

– **Definite Integrals**

1. Consider the two equivalent expresssions:

   * $\lim_{||p|| \to 0} \sum_{k=1}^{n} x_k^* e^{x_k^*} \Delta x_k$, where p is a partition of [1,4],
   * $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} x_k^* e^{x_k^*} \Delta x_k$, where the interval [1,4] is divided into n sub-intervals, the k'th interval has width = Δx_k,

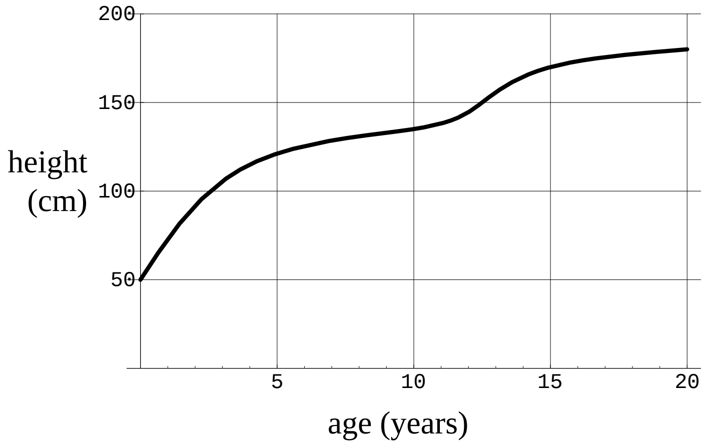
   and x_k^* is some number in the k'th interval. Express either of these equivalent statements as a definite integral. Do not attempt to evaluate the resulting definite integral.

- **Theorems**

1. True or False (write the word *True* or *False*)
   (a) All continous functions on an open interval are differentiable on that interval.
   (b) All continous functions on a closed interval are integrable on that interval.
   (c) If f is differentiable on (a,b) and continous on $[a,b]$ then there exists at least one number c in (a,b) such that $f'(c)(b-a) = f(b) - f(a)$.
2. State one version of the Fundemental Theorem of Calculus.
3. Let $F(x) = \int_0^x \cos(\frac{\pi}{2}t)\,dt$. Find $F'(x)$ and $F''(x)$.

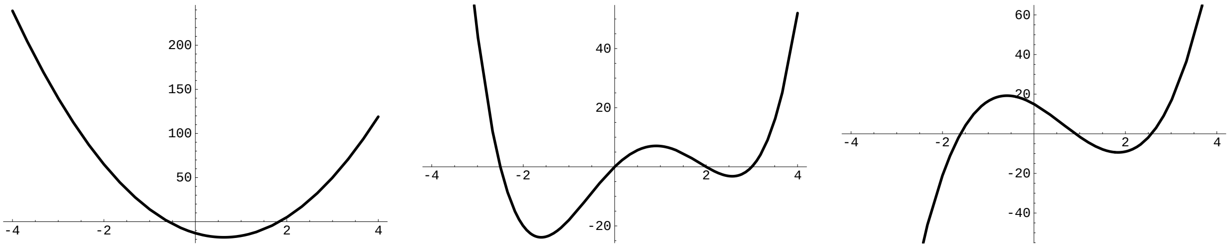
- **Analyzing a given graph**

1. The accompanying figure shows the graph of the height h in centimeters versus the age t in years of an individual from birth to age 20.



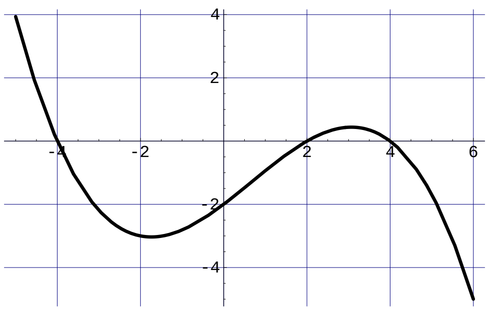
   (a) When is the growth rate the greatest?
   (b) Esitmate the growth rate at age 5.
   (c) At approximately what age between 10 and 20 is the growth rate the greatest? Estimate the growth rate at this age.

2. Below are the graphs of a function; $f(x)$, its first derivative; $f'(x)$, and its second derivative; $f''(x)$. Label them accordingly.

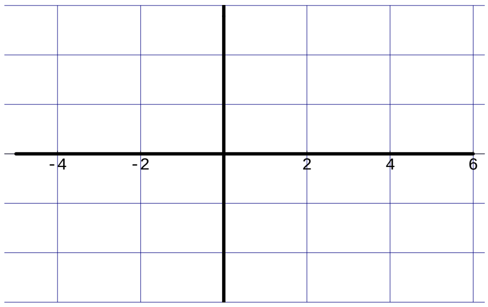


3. Suppose that f is a differentiable function. The graph of f', the derivative of f, over the interval [-5,6] is shown below.

Graph of $f'(x)$



Graph of $f''(x)$



(a) On what interval(s) is f increasing? Describe the evidence that is relevant. (A reasonable approximation to the interval(s) together with a clear explanation will suffice).

(b) On what interval(s) is the graph of f concave up? Describe the relevant evidence.

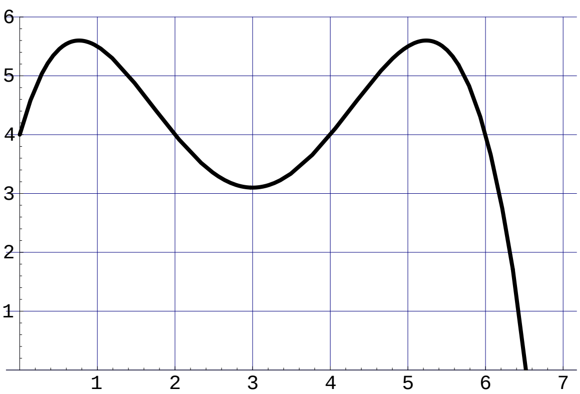
(c) Does f have a local (relative) maximum somewhere in the open interval (-5,6)? Describe the relevant evidence.

(d) Supose $f(0) = 12$. Write an equation for the line tangent to the graph of f at the point (0,12).

(e) Sketch the graph of f'', the second derivative of f. Describe two key features of your graph and explain how they are related to the graph shown above.

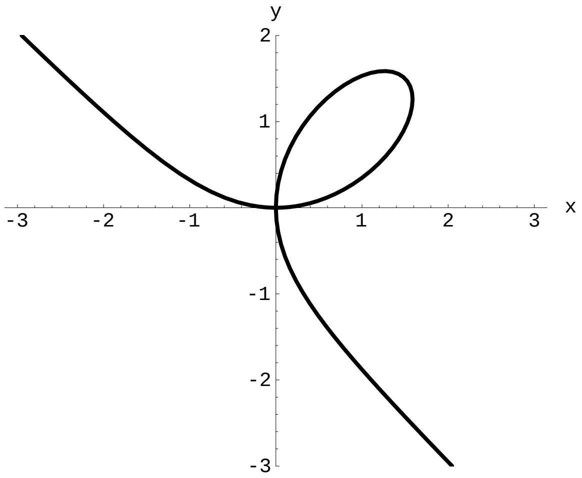
4. Consider the following graph where the anti-derivative function F of the function $f(x)$ is given by $F'(x) = f(x)$. Use the graph to evaluate $\int_2^6 f(x)\,dx$.

Graph of $F(x)$, where $F'(x) = f(x)$



- Below is a graph of the *Folium of Descartes* where y is defined implicitly by $x^3 + y^3 = 3xy$.

The Graph of $x^3 + y^3 = 3xy$



1. Use implicit differentiation to find an expression for $\dfrac{dy}{dx}$ at points on the curve where the tangent line is not vertical.
2. The point $(\frac{3}{2}, \frac{3}{2})$ is on this curve. What is the slope of the tangent line to the curve at this point?
3. Sketch the tangent line to the curve at the point $(\frac{3}{2}, \frac{3}{2})$ on the graph above.