

## Calculus II: Sample of Final Exam Questions.

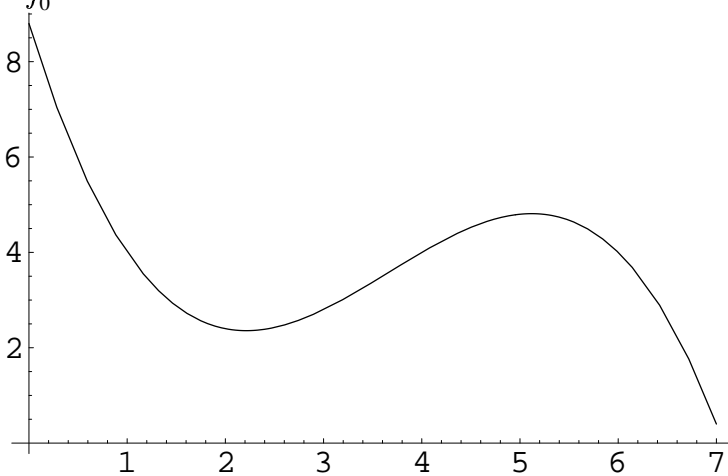
1. Express the following limit as a definite integral over the interval  $[0,2]$ , and evaluate the integral. Here,  $\Delta x_k$  is the width of the  $k$ 'th interval and  $x_k^*$  is some point in that interval. Use a formula from geometry to evaluate the limit.

$$\lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n \sqrt{4 - (x_k^*)^2} \Delta x_k =$$

2. The following limit equals the definite integral of some function  $g$  over the interval  $[5,12]$ . Your task is to write a definite integral that equals this limit.

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n (7 + (5 + k \cdot \Delta x)^3) \cdot \Delta x \quad \text{where} \quad \Delta x = \frac{7}{n}.$$

3. The following plot shows a graph of  $f$  over the interval  $[0,7]$ ; a few values of this function are listed in the table to the right of the plot. Describe a strategy for using this information to compute a Riemann sum that approximates the value of  $\int_0^6 f(x) dx$ , then use your strategy to calculate such an approximation.



$x$	$f(x)$
0	8.8
1	4.0
2	2.4
3	2.8
4	4.0
5	4.8
6	4.0
7	0.4

4. Consider the region enclosed between  $y = x(1 - x)$  and the  $x$ -axis.
- Sketch the enclosed region.
  - What is a good method for determining the volume of the solid generated by rotating the region about the  $y$ -axis? Why would you prefer this method over some other.
  - Set up but do not evaluate the integral describing the volume in part (b).
  - Set up but do not evaluate the integral describing the volume generated by revolving the region around the  $x$ -axis.
5. Write an integral expression for the area of the region enclosed by  $y = x + 2$  and  $y = x^2$ .
6. Use the method of partial fractions to evaluate the indefinite integrals below.
- $\int \frac{1}{(x-1)(x+1)} dx.$
  - $\int \frac{1}{(x-1)(x^2+1)} dx.$
7. Evaluate the following definite integral by making the substitution  $x = \tan \theta$ .
- $$\int_0^1 \sqrt{1+x^2} dx$$
8. Using the previous problem as a guide, find the arc length of the curve  $y = x^2/2$  from  $x = 0$  to  $x = 2$ .

9. Here we investigate the improper integral  $\int_0^1 \ln x \, dx$ .

(a) What makes the above definite integral improper?

(b) Use integration by parts to show that  $\int \ln x \, dx = x \ln(x) - x + C$ .

(c) Use part (b) to express the improper integral  $\int_0^1 \ln x \, dx$  as a limit.

(d) Does the improper integral converge or diverge. If it converges, to what does it converge. Hint: you will need to apply L'Hopital's rule to the expression  $\lim_{x \rightarrow 0^+} x \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x}$ .

10. At time  $t = 0$ , a tank contains 25 ounces of salt dissolved in 50 gal of water. Then brine containing 4 ounces of salt per gallon of brine is allowed to enter the tank at a rate of 2 gal/min and the mixed solution is drained from the tank at the same rate. Let  $y(t)$  denote the amount of salt in the tank at time  $t$ .

(a) Set up the initial value problem describing the rate of change in  $y$  with respect to time.

(b) Solve the initial value problem for  $y(t)$ .

(c) As  $t \rightarrow \infty$  what does the **concentration** of salt in the tank tend towards.

11. Use the integral test to determine whether the following series' converge. Be sure to justify the test by confirming the function used in the test satisfies the necessary conditions.

(a)  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

(b)  $\sum_{k=1}^{\infty} \frac{5}{e^{(k/2)}}$

12. In this problem we investigate the geometric series containing a variable  $x$ .

$$\sum_{n=0}^{\infty} \left(x - \frac{3}{2}\right)^n$$

(a) Does the series converge if  $x = 1$ ? Does it converge if  $x = 3$ ? Explain your answers.

(b) For what interval of values for  $x$  does the series converge? Explain your reasoning.

(c) Suppose  $w$  is a number in the convergence interval you found above. Write a simple expression that computes the sum of the infinite series when  $x = w$ .

13. Show that the alternating harmonic series (below) converges. State the test you use and show all parts of the test.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

14. Determine whether the series converges. Justify your answer.

(a)  $\sum_{k=1}^{\infty} \frac{3^k}{k!}$

(b)  $\sum_{k=1}^{\infty} \frac{1}{5 + 4^{-k}}$

15. Which of the functions graphed below is most likely to have  $p(x) = 1 - x + 2x^2$  as its second-order Maclaurin polynomial. Explain your reasoning.



16. Here we use the Remainder Estimation Theorem to reach some conclusions regarding a Taylor polynomial and the Taylor Series for  $\sin x$  about the point  $x_0 = \pi/2$ .

- Write the first 3 nonzero terms of the Taylor series for  $\sin x$  about  $x_0 = \pi/2$ .
- Find an upper bound on the error that results from approximating  $\sin x$  with these first three terms over the interval  $x = 0$  to  $x = \pi$ .
- Construct the Taylor Series for  $\sin x$  about  $x_0 = \pi/2$  and express your answer in summation notation.
- Prove that the above Taylor Series converges to  $\sin x$  over the interval  $x = 0$  to  $x = \pi$ .
- For what values of  $x$  does the Taylor series from part (c) converge to  $\sin x$ ? Justify your answer.

17. We already know the differentiation and integration formulas for  $\sin x$  and  $\cos x$ . In this problem we arrive at these formulas based on the Maclaurin Series for these functions and the differentiation and integration formulas for polynomials.

- Construct the Maclaurin series for  $\sin x$ . Find the radius of convergence for this series.
- Construct the Maclaurin series for  $\cos x$ . Find the radius of convergence for this series.
- Differentiate the Maclaurin series for  $\sin x$  term by term and verify the differentiation formula

$$\frac{d}{dx}[\sin x] = \cos x.$$

- Integrate the Maclaurin series for  $\sin x$  term by term and verify the integration formula

$$\int \sin x \, dx = -\cos x + C.$$

18. Use the fact that an antiderivative of  $\frac{1}{1+x^2}$  is  $\tan^{-1} x$  to derive the Maclaurin Series for  $\tan^{-1} x$ . What is the radius of convergence for this series. Use this series to find the fifth derivative of  $\tan^{-1} x$  evaluated at  $x = 0$ .

19. In this problem we investigate arclength and tangent lines of a circle defined in polar coordinates by  $r = 4 \cos \theta$ .

- How is this circle described in rectangular coordinates and sketch it.
- Find the slope of the tangent line to the circle for an arbitrary value of  $\theta$ .
- Sketch the tangent lines and state the slope for  $\theta = \pi/4$  and  $\theta = \pi/6$ .
- What range of  $\theta$  values result in one tracing of the circle.
- Verify that the circumference of the circle is  $4\pi$  by evaluating the integral describing the arclength.

These are questions specific to Section 1

20. A colony of the bacterium *E. coli* grows **continuously** at a relative rate of 80 percent when placed in a nutrient culture. Let  $y = y(t)$  be the number of cells that are present after 100 cells are placed in the culture.
- Find an initial value problem whose solution is  $y(t)$ .
  - Find a formula for  $y(t)$ , where all constants are assigned a numerical value.
  - How long does it take for the number of cells to double.
21. Express the following statements as a differential equation. Be sure to state what each variable stands for and whether any constants are positive or negative.
- The rate at which the temperature of a warming turkey increases in an oven is proportional to the difference between the oven temperature and the temperature of the turkey.*
  - The amount of a drug that is present in the blood stream tends to decrease at a rate that is proportional to the amount present.*
22. Solve the differential equation:  $\frac{\sqrt{1+x^2}}{1+y} \frac{dy}{dx} = -x$ .
23. Solve the initial value problem:  $x \frac{dy}{dx} + y = \sin(x), \quad y\left(\frac{\pi}{2}\right) = 1$ .