Homework #2 Due: Friday February 15

Math 471

1. Consider the bisection method which brackets a root r of $f(x) = 0$ with an initial interval of $[a_0, b_0]$, and $c_n = (a_n + b_n)/2$. Given $\varepsilon > 0$, find N such that $|c_n - r| < \varepsilon$ for all $n > N$.

2. Recall that for a simple zero r of f: $f(r) = 0$ and $f'(r) \neq 0$, and an initial x_0 chosen *close enough* to r, the sequence generated by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g(x_n)$$

generates a sequence of x_n's satisfying

$$|e_{n+1}| \to K_1 |e_n|^2 \quad \text{where} \quad K_1 = \left| \frac{f''(r)}{2f'(r)} \right|,$$

where $e_n = x_n - r$.

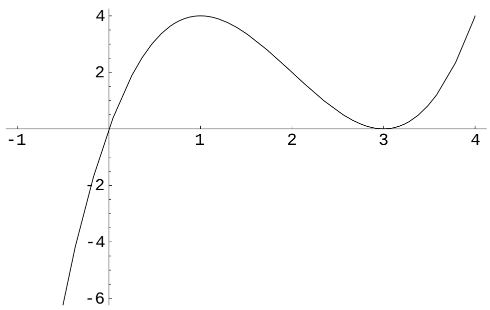
Similarly, we showed that

$$|e_{n+1}| \to K_2 |e_n|^2 \quad \text{where} \quad K_2 = \left| \frac{g''(r)}{2} \right|.$$

Show that $K_1 = K_2$ provided $f'''(x)$ exists on some interval about r.

3. Do one of the following problems.

   (a) Consider the function $f(x) = x(x-3)^2$ graphed below.



For these problems I want you to print up a graph of $\ln|e_{n+1}|$ -vs- $\ln|e_n|$, where $e_n = x_n - r$ which illustrates the slope of the relationship verifying:

   i. Newton's method converges quadratically to $r = 0$ for $x_0 = 0.5$
   ii. Newton's method converges linearly to $r = 3$ for $x_0 = 2$.
   iii. The secant method converges to $r = 0$ for $x_0 = 0.5$ and $x_1 = 0.25$ with order of convergence approximately 1.62.
   iv. Newton's method adapted for a double root converges quadratically to $r = 3$ for $x_0 = 2$.

(b) Cardiac Output may be approximated by Stretching or Compressing the function

$$Q(t) = \sin^n(\pi t)\ \cos(\pi t - \phi)$$

where $n \approx 13$ (odd) and $\phi \approx \pi/10$. Figure 1 shows a graph of this function with $n = 13$ and $\phi = \pi/10$. One value of particular interest to cardiologists is the peak-to-mean flow ratio. To calculate this from the approximating function $Q(t)$, you need first, to find the time (t^*) at which the peak outflow occurs ($t^* \approx 0.45$). Therefore we take the derivative and set it equal to zero. We want to know where does

$$Q'(t) = \pi \sin^{n-1}(\pi t)\,[n\ \cos(\pi t)\cos(\pi t - \phi) - \sin(\pi t)\sin(\pi t - \phi)] = 0$$

Since the first factor of $Q'(t) = 0$ at $t = 0$ and $t = 1$ we really want to find where

$$f(t) = n\ \cos(\pi t)\cos(\pi t - \phi) - \sin(\pi t)\sin(\pi t - \phi) = 0$$

and specifically find the zero near $t = 0.45$.

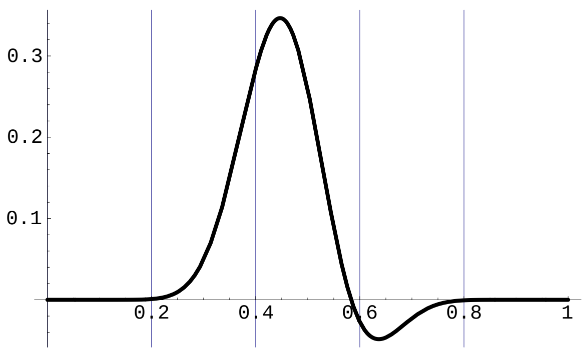


Figure 1: $Q(t)$

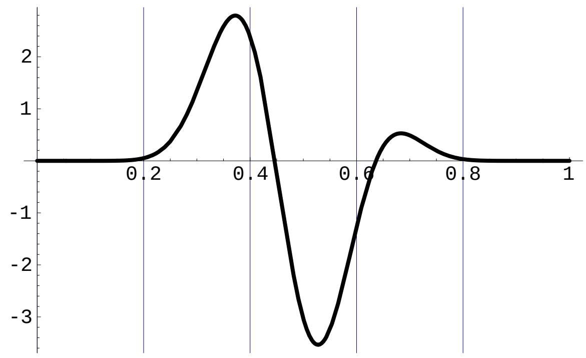


Figure 2: $Q'(t)$

Your Assignment: Use any root finding technique to fill in the charts below, using 4 decimal places in your approximations for t^*. **Note:** It is possible that your values of t^* will be constant down a column.

for $n = 13$

ϕ	t^*
$\pi/12$	
$\pi/10$	
$\pi/8$	
$\pi/6$	

for $\phi = \pi/10$

n	t^*
9	
11	
13	
15	