MATH 220 Exam 2 Answers

1. Assume A is an n x n matrix and $\det(A) = 3$. *(10 pts)*

   T = The statement is always true.

   F = The statement is not always true.

   (a) **T** **F** $\rho(A) = n$ Thrm 7 (4.7)

   (b) **T** **F** The rows of A form a **linearly dependent** set of vectors.

   (c) **T** **F** The columns of A form a **linearly independent** set of vectors. Thrms 5 & 6 From 4.5

   (d) **T** **F** The null space of A, denoted N_A, has dimension 1. No $\nu(A) + \rho(A) = n$ $\nu(A) = 0$

   (e) **T** **F** The columns in A form a basis for R^n. They span $\mathbb{R}^n$ because $\dim(C_A) = n$ they are lin. ind.

2. Only one answer is correct, circle it. Circle *don't know* and you lose only 3 out of 5 points.

   - If A is a 4 x 4 matrix and $\det(A) = 0$ then the minimum possible value for $\nu(A)$ is Know $\rho(A) < 4$ $\therefore$ $\nu(A) > 0$

     (a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) *don't know*

   - If $B = \begin{pmatrix} -3 & -3 & 4 & 5 \\ 0 & 2 & -8 & 5 \\ 0 & 0 & 0 & 5 \end{pmatrix}$, then the rank of B is 3-pivots

     (a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) *don't know*

3. You are given a vector space V and a non-empty subset of V called H. You must determine whether H is a subspace of V. If it is, prove that H satisfies the properties of a subspace. If it is not, demonstrate that it does not satisfy one of the properties. *(20 pts)*

   (a) Vector Space: $V = M_{22}$, the set of all 2 x 2 matrices.

   $$H = \left\{ A \in M_{22} \; : \; A = \begin{pmatrix} a & 1+a \\ 0 & 0 \end{pmatrix} \right\}$$

   Bad $A+B \notin H$

   let $A = \begin{pmatrix} a & 1+a \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} b & 1+b \\ 0 & 0 \end{pmatrix}$ $\Rightarrow$ $A+B = \begin{pmatrix} a+b & 2+(a+b) \\ 0 & 0 \end{pmatrix}$

   scalar mult. $\alpha A = \begin{pmatrix} \alpha a & \alpha + \alpha a \\ 0 & 0 \end{pmatrix}$ need $\alpha = 1$ Bad $\alpha A \notin H$

   Not A Subspace.

   (b) Vector Space: $V = M_{22}$, the set of all 2 x 2 matrices.

   $$H = \{ A \in M_{22} \; : \; Ax = 0 \} \text{ where } x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   if $A \in H$ & $B \in H$ Then $(A+B)x = Ax + Bx = 0$ So $A+B \in H$ ✓

   if $A \in H$ α = scalar $(\alpha A)x = \alpha(Ax) = \alpha 0 = 0$ so $\alpha A \in H$ ✓

   Is A Subspace.

4. Find a basis in R^3 for the solution space of the given homogeneous system. *(20 pts)*

$$x - y - z = 0$$
$$2x - y + z = 0$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right] R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \quad R_1 \to R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

$x + 2z = 0$
$y + 3z = 0$

If z is arbitrary
$x = -2z, \quad y = -3z$

Solutions
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ -3z \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}}_{\text{Basis}} z$$

$$\text{Basis} = \left\{ \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} \right\}$$

5. Determine whether the given vectors span R^3. *(10 pts)*

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, & \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, & \begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix} \end{array}$$

can any $v \in \mathbb{R}^3$ be written as

$v = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 3 & 5 \\ 0 & 4 & 6 & 10 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = v$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & x \\ 0 & 2 & 3 & 5 & y \\ 0 & 4 & 6 & 10 & z \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & x \\ 0 & 2 & 3 & 5 & y \\ 0 & 0 & 0 & 0 & z - 2y \end{array}\right]$$

Need $z - 2y = 0$

That means $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is not any vector in $\mathbb{R}^3$.

Do they span R^3?
No

Could say
$\rho(A) = 2$
$\dim(C_A) = 2$
Know $\dim(\mathbb{R}^3) = 3$

So the columns can't span all of $\mathbb{R}^3$.

6. Prove that if $v_1, v_2, \ldots, v_n$ span R^n then $v_1, v_2, \ldots, v_n, v_{n+1}$ also span R^n. That is, the addition of one (or more) vectors to a spanning set yields another spanning set. *(10 pts)*

given $v_1, v_2, \ldots v_n$ span $\mathbb{R}^n$ $\Leftrightarrow$ any vector in $\mathbb{R}^n$ $v = c_1 v_1 + c_2 v_2 + \cdots c_n v_n$ for some $c_1, c_2, \ldots c_n$,

and $\star$ $v = c_1 v_1 + c_2 v_2 + \ldots c_n v_n + c_{n+1} v_{n+1}$

where $c_{n+1} = 0$

by $\star$ any vector in $\mathbb{R}^n$ can be written as a linear combination of $v_1, v_2, \ldots v_{n+1}$

This is the definition of span.

and $\{v_1, v_2, \ldots v_{n+1}\}$ also span $\mathbb{R}^n$