Bonus problem for Calculus III
Kepler's Laws of Planetary Motion

Johannes Kepler (1571-1630) is now chiefly remembered for discovering the three laws of planetary motion that bear his name published in 1609 and 1619. In this problem we verify these three laws.

Assume each planet moves in an orbit given by the vector-valued function $\mathbf{r}(t)$. Let $r(t) = ||\mathbf{r}(t)||$, let G represent the universal gravitational constant, let M represent the mass of the sun, and let m represent the mass of the planet.

1. Prove that $\mathbf{r} \cdot \mathbf{r}' = r\dfrac{dr}{dt}$.

2. Using Newton's Second Law, $\mathbf{F} = m\mathbf{a}$, and Newton's Second Law of Gravitation, $\mathbf{F} = -(GmM/r^3)\mathbf{r}$, show that $\mathbf{a}$ and $\mathbf{r}$ are parallel, and that

   $\mathbf{r}(t) \times \mathbf{r}'(t) = \mathbf{L}$

   is a constant vector. Hence, $\mathbf{r}(t)$ moves in a *fixed plane*, orthogonal to $\mathbf{L}$.

3. Prove that $\dfrac{d}{dt}\left[\dfrac{\mathbf{r}}{r}\right] = \dfrac{1}{r^3}\left\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\right\}$.

4. Show that

   $\dfrac{\mathbf{r}'}{GM} \times \mathbf{L} - \dfrac{\mathbf{r}}{r} = \mathbf{e}$

   is a constant vector.

5. Prove that $\mathbf{e}$ is in the same plane as $\mathbf{r}$ and hence in the same plane as $\mathbf{r}'(t)$.

6. Situate $\mathbf{r}$ and $\mathbf{e}$ in the xy-plane with the sun as the origin, $\mathbf{e}$ is along the positive x-axis, and θ is the angle between $\mathbf{e}$ and $\mathbf{r}$.

   Prove

   $$r = \frac{||\mathbf{L}||^2/GM}{1 + e\cos\theta},$$

   where $e = ||\mathbf{e}||$.

7. Prove Kepler's First Law: Each planet moves in an elliptical orbit with the sun as a focus.

8. Assume that the elliptical orbit

   $r = \dfrac{ed}{1 + e\cos\theta}$

   is in the xy-plane, with $\mathbf{L}$ along the z-axis. Prove that

   $||\mathbf{L}|| = r^2\dfrac{d\theta}{dt}$.

9. Prove Kepler's Second Law: Each ray from the sun to a planet sweeps out equal areas of the ellipse in equal times.

10. Prove Kepler's Third Law: The square of the period of a planet's orbit is proportional to the cube of the mean distance between the planet and the sun.

Ellipses in polar coordinates

In polar coordinates an ellipse can be expressed as

$$r = \frac{1}{1 + \epsilon \cos\theta}$$

where $0 < \epsilon < 1$.

- This ellipse has (0,0) as a focus.
- The other focus is at $\left(\frac{-2\epsilon}{1-\epsilon^2}, 0\right)$.
- The horizontal axis is the major axis with length $= \frac{2}{1-\epsilon^2}$
- The vertical axis is the minor axis with length $= \frac{2}{\sqrt{1-\epsilon^2}}$.
- In rectangular coordinates this ellipse is expressed as

$$\frac{\left(x + \frac{\epsilon}{1-\epsilon^2}\right)}{\frac{1}{(1-\epsilon^2)^2}} + \frac{y^2}{\frac{1}{1-\epsilon^2}} = 1$$

- This is a circle if $\epsilon = 1$.