Math 250, Section 1, TEST 4 **Answers**

1. Find the singular point of the given differential equation and classify it as regular or irregular. Justify your classification. *(10 pts)*

$$x^2y'' + 2y' + 3y = 0$$

The only singular point is at $x_o = 0$. Notice $\lim_{x\to 0} x\frac{2}{x^2} = \lim_{x\to 0}\frac{2}{x}$ is **not finite** and therefore the singular point is **irregular**. There is no need to check the second condition for a regular singular point.

2. Solve the following initial value problem. Assume $x > 0$. *(10 pts)*

$$x^2y'' - 5xy' + 9y = 0 \qquad y(1) = 3 \quad \text{and} \quad y'(1) = 2$$

Assume a solution of the form $y = x^r$. Finding y' and y'' leads to $x^r\left[r(r-1) - 5r + 9\right] = 0 \to (r-3)^2 = 0$ so $r = 3$ is a double root. Therefore the general solution is $y(x) = c_1x^3 + c_2x^3\ln(x)$. Now $y'(x) = 3c_1x^2 + 3c_2x^2\ln(x) + c_2x^3\frac{1}{x}$. So $y(1) = c_1$ and $y'(1) = 3c_1 + c_2$. Imposing the initial conditions gives $c_1 = 3$ and $c_2 = -7$. So $y(x) = 3x^3 - 7x^3\ln(x)$.

3. Find the inverse Laplace transform of the following functions. *(20 pts)*

(a) $F(s) = \dfrac{2e^{-2s}}{s^2-4} = e^{-2s}\dfrac{2}{s^2-4} = e^{-2s}\dfrac{2}{s^2-2^2}$

$f(t) = u_2(t)g(t-2)$ where $g(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2-2^2}\right\} = \sinh 2t$ Therefore, $f(t) = u_2(t)\sinh 2(t-2)$

(b) $F(s) = \dfrac{s+4}{s^2-2s+5}$

$F(s) = \dfrac{s+4}{(s-1)^2+4} = \dfrac{(s-1)+5}{(s-1)^2+4} = \dfrac{s-1}{(s-1)^2+4} + \dfrac{5}{2}\dfrac{2}{(s-1)^2+4}$. Therefore $f(t) = e^t\cos(2t) + \frac{5}{2}\sin(2t)$.

4. Find the Laplace transform of the following function. *(10 pts)*

$$f(t) = \begin{cases} 0 & t < 2 \\ (t-2)^3 & t \geq 2 \end{cases}$$

$f(t) = u_2(t)g(t-2)$ where $g(t) = t^3$. Therefore $F(s) = e^{-2s}G(s)$ where $G(s) = \mathcal{L}\left\{t^3\right\} = \frac{3!}{s^4}$.

5. Use the Laplace transform to solve the initial value problem *(20 pts)*

$$y'' + 2y' - 3y = 0, \qquad y(0) = 1, \quad y'(0) = 2$$

Taking the Laplace transform of the entire equation yields

$[s^2\mathbf{Y}(s) - sy(0) - y'(0)] + 2\,[s\mathbf{Y}(s) - y(0)] - 3\mathbf{Y}(s) = 0$

plugging in the initial conditions

$[s^2\mathbf{Y}(s) - s - 2] + 2\,[s\mathbf{Y}(s) - 1] - 3\mathbf{Y}(s) = 0$

$\mathbf{Y}(s)[s^2 + 2s - 3] = s + 4$

$\mathbf{Y}(s) = \frac{s+4}{s^2+2s-3} = \frac{s+4}{(s+3)(s-1)}$

partial fractions gives $\mathbf{Y}(s) = \frac{-1}{4}\left(\frac{1}{s+3}\right) + \frac{5}{4}\left(\frac{1}{s-1}\right)$.

Taking the inverse Laplace transform yields

$y(t) = \frac{-1}{4}e^{-3t} + \frac{5}{4}e^{t}$.

6. Find the solution of the initial value problem *(30 pts)*

$$y'' + 9y = u_2(t), \qquad y(0) = 1, \quad y'(0) = 2$$

Taking the Laplace transform of the entire equation yields

$[s^2\mathbf{Y}(s) - sy(0) - y'(0)] + 9\mathbf{Y}(s) = \frac{e^{-2s}}{s}$

plugging in the initial conditions

$[s^2\mathbf{Y}(s) - s - 2] + 9\mathbf{Y}(s) = \frac{e^{-2s}}{s}$

$\mathbf{Y}(s)(s^2 + 9) = 2 + s + \frac{e^{-2s}}{s}$

$\mathbf{Y}(s) = \frac{2}{s^2+9} + \frac{s}{s^2+9} + e^{-2s}\frac{1}{s(s^2+9)}$

Partial fractions on the second factor of the last term yields $\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$ where $A = 1/9$, $B = -1/9$, and $C = 0$.

So

$\mathbf{Y}(s) = \dfrac{2}{3}\left(\dfrac{3}{s^2+9}\right) + \dfrac{s}{s^2+9} + \dfrac{e^{-2s}}{9}\left[\dfrac{1}{s} - \dfrac{s}{s^2+9}\right]$

and taking the inverse Laplace transform yields

$y(t) = \dfrac{2}{3}\sin(3t) + \cos(3t) + \dfrac{u_2(t)}{9}h(t-2)$ where $h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+9}\right\} = 1 - \cos(3t)$

So

$y(t) = \dfrac{2}{3}\sin(3t) + \cos(3t) + \dfrac{u_2(t)}{9}\left[1 - \cos 3(t-2)\right]$