

1. Solve the following initial value problems.

(30 pts)

(a) $y' - \alpha y = e^{\alpha t}$, $y(0) = b$.

This is a linear equation with variable coefficients where $p(t) = -\alpha$ and $g(t) = e^{\alpha t}$. Therefore $u(t) = e^{\int p(t) dt} = e^{\int -\alpha dt} = e^{-\alpha t}$. Upon multiplying both sides by $u(t)$ we get $[uy]' = ug$ so $uy = \int ugd t = \int e^{-\alpha t} e^{\alpha t} dt = \int dt = t + c$. Now we have $uy = e^{-\alpha t} y = t + c$, solving for y we get $y = e^{\alpha t}(t + c)$. Solve for c from $y(0) = b$ and you get $c = b$ so the solution is

$$y = e^{\alpha t}(t + b).$$

(b) $y' \pm y^2 \text{ trig}(x) = 0$, $y(0) = \frac{1}{b}$

The \pm , b , and trig function depends on which test you took. In either case, the equation is separable. You can rewrite it as $y' = \mp y^2 \text{ trig}(x) \rightarrow \frac{dy}{y^2} = \mp \text{trig}(x) dx$. Now integrating both sides yields $\frac{-1}{y} = \mp \int \text{trig}(x) dx$. At this point you integrate the correct trig function (sin or cos), include a constant of integration, and solve for y . You then get the constant of integration from the condition $y(0) = \frac{1}{b}$.

2. Solve the following differential equations. You need not solve y explicitly in terms of x .

(25 pts)

(a) $\frac{dy}{dx} = \frac{x + e^x}{y - e^{-y}}$ or $\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$

This is a separable equation

(1) $\rightarrow (y - e^{-y}) dy = (x + e^x) dx \rightarrow \frac{y^2}{2} + e^{-y} = \frac{x^2}{2} + e^x + c$ (done)

(2) $\rightarrow (y + e^y) dy = (x - e^{-x}) dx \rightarrow \frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + c$ (done)

(b) $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3) \frac{dy}{dx} = 0$ (number 3 section 2.6)

This is an exact equation where $M(x, y) = 3x^2 - 2xy + 2$ and $N(x, y) = 6y^2 - x^2 + 3$. Notice $M_y = -2x = N_x$. Now we know that $\Psi_x = M$ so we integrate $M(x, y)$ with respect to x and get $\Psi(x, y) = x^3 - x^2y + 2x + h(y)$. We also know that $\Psi_y = N(x, y)$ this gives $-x^2 + h'(y) = 6y^2 - x^2 + 3$ therefore $h'(y) = 6y^2 + 3$. Integrating $h'(y)$ with respect to y yields $h(y) = 2y^3 + 3y + c_1$ and we'll let $c_1 = 0$. This gives $\Psi(x, y) = x^3 - x^2y + 2x + (2y^3 + 3y)$. The solution is then

$$x^3 - x^2y + 2x + 2y^3 + 3y = c$$

3. A tank initially contains V liters of pure water. A mixture containing a concentration of γ g/liter of salt enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Answer the following questions. Keep in mind that the answers will depend on the unknown parameter γ . (25 pts)

(a) Let $Q(t)$ denote the grams of salt in the tank at any time t . Derive a differential equation which has $Q(t)$ as the solution. **Include an initial value.**

$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$. The rate in is 2γ and the rate out is $2Q(t)/V$. So

$$\frac{dQ}{dt} = 2\gamma - 2\frac{Q}{V}$$

Since the tank starts off with pure water we get the initial condition $Q(0) = 0$.

(b) Solve the initial value problem derived in part (a).

Rewrite the result from part a as

$$\frac{dQ}{dt} + \frac{2}{V}Q = 2\gamma.$$

This is linear with variable coefficients where $p(t) = \frac{2}{V}$ and $g(t) = 2\gamma$. We get $u = e^{\frac{2t}{V}}$ and then $uQ = \int ug = 2\gamma \int e^{\frac{2t}{V}} = \gamma V e^{\frac{2t}{V}} + c$. So $Q(t) = \gamma V + ce^{-\frac{2t}{V}}$. Since $Q(0) = 0$, we get $\gamma V + c = 0 \rightarrow c = -\gamma V$.

$$Q(t) = \gamma V - \gamma V e^{-\frac{2t}{V}} = \gamma V (1 - e^{-\frac{2t}{V}})$$

(c) What is the limiting value of Q as $t \rightarrow \infty$?

Letting $t \rightarrow \infty$ we can see from the above equation that the exponential goes to zero and $Q \rightarrow \gamma V$. You can deduce this from assuming that the concentration in the tank will get closer to the incoming concentration and so the amount in the tank will get close to the incoming concentration times the volume.

4. Determine, **without solving the problem**, an interval in which the solution of the given differential equation is certain to exist. (10 pts)

$$(t - 5)y' + \sqrt{t}y = e^{2t}, \quad y(1) = 2$$

5. Consider the differential equation $\frac{dy}{dt} = (y - a)(b - y)$. (10 pts)

(a) Make a rough sketch of the graph of y' versus y .

(b) Suppose $y(0) = (a + b)/2$. **Without solving the differential equation**, determine what happens to y as t gets large. Explain your reasoning.

At $y(0) = (a + b)/2$ you can see that y' is positive so y will increase. As y increases y' will decrease but remain positive. So y will increase towards b (but never get there) as $t \rightarrow \infty$. I.e.: $\lim_{t \rightarrow \infty} y = b$.

(c) Suppose $y(0) = b + 1$. **Without solving the differential equation**, determine what happens to y as t gets large. Explain your reasoning.

At $y(0) = b + 1$ you can see that y' is negative so y will decrease. As y decreases y' will tend to zero but remain negative. So y will decrease towards b (but never get there) as $t \rightarrow \infty$. I.e.: $\lim_{t \rightarrow \infty} y = b$.