Math 250, Section 1, TEST 2 **Answers**

1. Consider the differential equation: $y'' + 5y' + 6y = 0$ (*15 pts)*

   The characteristic equation is $r^2 + 5r + 6 = 0 \to (r+3)(r+2) = 0 \to r_1 = -3$ and $r_2 = -2$.

   (a) Find the *general solution* of the differential equation. $y = c_1 e^{-3t} + c_2 e^{-2t}$

   (b) Find the solution satisfying the initial conditions $y(0) = 0$ and $y'(0) = -3$.
   First we need $y'(t) = -3c_1 e^{-3t} - 2c_2 e^{-2t}$. So we have $y(0) = c_1 + c_2 = 0$ and $y'(0) = -3c_1 - 2c_2 = -3$. The first equation gives $c_1 = -c_2$. Putting this into the second equation gives $c_2 = -3$. Back to the first gives $c_1 = 3$. So $y = 3e^{-3t} - 3e^{-2t}$

2. Consider the differential equation: $y'' - 4y' + 4y = 0$ (*15 pts)*

   The characteristic equation is $r^2 - 4r + 4 = 0 \to (r-2)^2 = 0 \to r_1 = r_2 = 2$.

   (a) Find the *general solution* to the differential equation.$y = c_1 e^{2t} + c_2 t e^{2t} = (c_1 + c_2 t)e^{2t}$

   (b) Determine the possible behaviors of any non-zero solution as $t \to \infty$. Show work that justifies your answer. If $c_2 = 0$, $y \to \pm\infty$ depending on the sign of c_1. Otherwise $y \to \pm\infty$, depending on the sign of c_2. In general the solution goes to positive or negative infinity.

3. Consider the differential equation: $y'' + 2y' + 10y = 0$ (*15 pts)*

   The characteristic equation is $r^2 + 2r + 10 = 0$. The quadratic formula the gives $r = -1 \pm 2i$.

   (a) Find the *general solution* to the differential equation. $y = e^{-t}\left(c_1 \cos 2t + c_2 \sin 2t\right)$

   (b) Describe the behavior of any solution as $t \to \infty$. Show work that justifies your answer. Since $e^{-t} \to 0$ as $t \to \infty$, and the second factor is bounded, all solutions tend to 0.

4. If y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, prove that $c_1\, y_1$ and $c_2\, y_2$ are also linearly independent, provided that neither c_1 nor c_2 is zero. **Hint** Express the wronskian of the second set of solutions in terms of the wronskian of the first. (*10 pts)*

   $W[c_1 y_1, c_2 y_2] = c_1 c_2 y_1 y_2' - c_1 c_2 y_1' y_2 = c_1 c_2 (y_1 y_2' - y_1' y_2) = c_1 c_2 W[y_1, y_2]$

   The wronskian of y_1 and y_2 (the second factor on the right of the above equation; $W[y_1, y_2]$) is never zero because they are linearly independent and solutions of the differential equation. Since c_1 and c_2 are nonzero by hypothesis, the right hand term is never zero. This implies that the wronskian of $c_1 y_1$ and $c_2 y_2$ (the left hand side above) is never zero. Therefore $c_1 y_1$ and $c_2 y_2$ are linearly independent.

5. Find a particular solution **Y** of the nonhomogeneous equation; $y'' - 2y' - 3y = 6e^{2t}$. (*15 pts)*

   The characteristic equation of the homogeneous equation is $r^2 - 2r - 3 = 0 \to (r-3)(r+1) = 0 \to r_1 = 3$ and $r_2 = -1$. So a fundamental set of solutions to the homogeneous equation is $y_1 = e^{3t}$, $y_2 = e^{-t}$.

   method of undetermined coefficients: Look for a solution of the form $Y = Ae^{2t}$. This is not a solution of the homogeneous equation so we don't have to multiply it by t. So $Y' = 2Ae^{2t}$, and $Y'' = 4Ae^{2t}$. Plugging Y, Y', and Y'' into the differential equation, cancelling the exponential term, and simplifying yields $4A - 4A - 3A = 6 \to A = -2$. So the particular solution is $Y = -2e^{2t}$.

   variation of parameters (formula) $W[y_1, y_2] = -4e^{2t}$, $g(t) = 6e^{2t}$, so $g/W = -3/2$.

$$\mathbf{Y} = -y_1 \int \frac{y_2\, g}{W} dt + y_2 \int \frac{y_1\, g}{W} = -e^{3t}(-3/2) \int e^{-t} dt + e^{-t}(-3/2) \int e^{3t} dt = -\frac{3}{2}e^{2t} - \frac{1}{2}e^{2t} = -2e^{2t}$$

6. Give an example of a non-homogeneous differential equation where the *method of undetermined coefficients* cannot be used and we would be forced to use the gruelling method of *variation of parameters*. *(10 pts)*

   Any differential equation of the form $y'' + p(t)y' + q(t)y = g(t)$ with either of the following properties

   (a) $g(t)$ is not a product or sum of polynomial, exponential, sine, or cosine functions.
   Ex: $y'' + y' + 2y = \tan(t)$.

   (b) $p(t)$ and/or $q(t)$ not constant.
   Ex: $y'' + ty' + y = e^t$.

7. Determine the **form** of the particular solution, **Y**, to the given differential equations. **Do not try to solve for the particular solution.** Note: This is the second step involved in *the method of undetermined coefficients*. Your answers should contain one or more undetermined coefficients. *(20 pts)*

   (a) $y'' - y' - 2y = 2e^{-t}$
   Seek a solution of the form $\mathbf{Y} = Ae^{-t}$ to match the nonhomogeneous term. But first, the characteristic equation is $r^2 - r - 2 = 0 \rightarrow (r-2)(r+1) = 0 \rightarrow r_1 = 2$, and $r_2 = -1$. So a fundamental set of solutions to the homogeneous equation is $y_1 = e^{2t}$, $y_2 = e^{-t}$. Notice, the suggested form of the particular solution is a solution to the homogeneous equation. Therefore we multiply the original form by t to get: $\mathbf{Y} = Ate^{-t}$.

   (b) $y'' - y' - 2y = 3\sin(2t)$
   Seek a solution of the form $\mathbf{Y} = A\cos 2t + B\sin 2t$. Since we found the characteristic equation and fundamental solutions in part (a) we know that this supposed form is not a solution of the homogeneous equation so no multiplication by t is necessary. $\mathbf{Y} = A\cos 2t + B\sin 2t$.

   (c) $y'' - y' - 2y = 7t^2$
   Seek a solution in the form of a polynomial of same degree as the nonhomogeneous term: $\mathbf{Y} = At^2 + Bt + C$. Again, this is not a solution of the homogeneous equation so no additional multiplication by t is necessary.$\mathbf{Y} = At^2 + Bt + C$.

   (d) $y'' + 4y = \cos(2t)$
   Seek a solution of the form $\mathbf{Y} = A\cos(2t) + B\sin(2t)$. But first, the characteristic equation is $r^2 + 4 = 0 \rightarrow r^2 = -4 \rightarrow r = \pm 2i$. So a fundamental set of solutions to the homogeneous equation is $y_1 = \cos 2t$, $y_2 = \sin 2t$. Notice, the suggested form of the particular solution is a solution to the homogeneous equation. Therefore we multiply the original form by t to get: $\mathbf{Y} = t\ [A\cos(2t) + Bsin(2t)]$.