Math 250, Section 1, TEST 3 **Answers**

1. Consider the nonhomogeneous differential equation: *(20 pts)*

$$y^{iv} - 2y'' + y = e^{-t} + e^{5t} + \cos t$$

where y^{iv} represents the fourth derivative of y.

(a) Find the *general solution* of the corresponding *homogenous equation.* The general solution to $y^{iv} - 2y'' + y = 0$ has the form e^{rt} where r satisfies the equation $r^4 - 2r^2 + 1 = 0$. This is a quadratic in r^2. Ie, let $v = r^2$ and this equation becomes $v^2 - 2v + 1 = (v-1)^2 = 0$ which has a repeated root $v = 1$ and therefore $r^2 = v \rightarrow r = \pm 1$. Each of these is repeated. Therefore the general solution to the homogenous equation is $y = ae^t + bte^t + ce^{-t} + dte^{-t}$

(b) Find a suitable form for a *particular solution* $\mathbf{Y}(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants. The nonhomogenous term e^{-t} suggests a particular solution of the form $\mathbf{Y}_1 = Ae^{-t}$ but this is a solution of the homogenous equation so you try $\mathbf{Y}_1 = Ate^{-t}$ but this also is a solution to the homogenous equation so you multiply by t one more time to get $\mathbf{Y}_1 = At^2e^{-t}$ The nonhomogenous term e^{5t} suggests a particular solution of the form $\mathbf{Y}_2 = Be^{5t}$. The nonhomogenous term $\cos(t)$ suggests a particular solution of the form $\mathbf{Y}_3 = C\sin t + D\cos t$. Finally the particular solution $\mathbf{Y}$ is the sum of these

$$\mathbf{Y} = At^2e^{-t} + Be^{5t} + C\sin t + D\cos t.$$

2. Find the radius of convergence (ρ) of the given power series. *(15 pts)*

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}(x-1)^n$ Use the ratio test to get

$$L = \lim_{n\to\infty} \left| \frac{(n+1)(x-1)^{(n+1)}}{2^{n+1}} \; \frac{2^n}{n(x-1)^n} \right| = \frac{|x-1|}{2} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x-1|}{2}$$

Therefore we need $\frac{|x-1|}{2} < 1 \iff |x-1| < 2$ so $\rho = 2$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Use the ratio test to get

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \; \frac{n}{x^n} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0$$

So $L < 1$ regardless of x and the power series converges for all x. This means $\rho = \infty$.

3. The differential equation $y'' - xy' - y = 0$ has two solutions defined by *(20 pts)*

$$\begin{aligned} y_1(x) &= 1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} + \frac{x^6}{2 \cdot 4 \cdot 6} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} \\ y_2(x) &= x + \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \frac{x^7}{3 \cdot 5 \cdot 7} + \ldots = x + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1)} \end{aligned}$$

(a) Verify that these two functions are linearly independent. It is easily shown that $y_1(0) = 1$ and $y_2(0) = 0$. Furthermore $y_1'(0) = 0$ and $y_2'(0) = 1$ so the Wronskian of y_1 and y_2 evaluated at $x = 0$ is 1. Because y_1 and y_2 are solutions to the above differential equation their Wronskian is never zero and must be linearly independent.

(b) Find the value of the constants a_0 and a_1 in the general solution

$$y = a_0\, y_1 + a_1\, y_2$$

which satisfy the initial conditions $y(0) = \frac{1}{2}$ and $y'(0) = 3$. Based on the same argument above, $y(0) = a_o$ and $y'(0) = a_1$. Therefore, $a_0 = 1/2$ and $a_1 = 3$.

4. Solve the differential equation $y'' + xy' + 2y = 0$. by means of a power series about the point $x_o = 0$. *(30 pts)*

Assume $y = \sum_{n=0}^{\infty} a_n x^n$. This yields $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Plugging these into the differential equation yields

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n &= 0 \end{aligned}$$

Taking the first term out of the first and third sums yields the **recurrence relation**

$(n+2)(n+1)a_{n+2} + n a_n + 2a_n = 0$ **or** $a_{n+2} = \dfrac{-a_n}{n+1}$

So a_0 and a_1 are arbitrary.

$a_2 = \frac{-a_0}{1}$, $a_4 = \frac{-a_2}{3} = \frac{a_0}{1 \cdot 3}$, $a_6 = \frac{-a_4}{5} = \frac{-a_0}{1 \cdot 3 \cdot 5}$ and $a_{2n} = \frac{(-1)^n a_0}{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)}$ for $n \geq 1$.

$a_3 = \frac{-a_1}{2}$, $a_5 = \frac{-a_3}{4} = \frac{a_1}{2 \cdot 4}$, $a_7 = \frac{-a_5}{6} = \frac{-a_1}{2 \cdot 4 \cdot 6}$ and $a_{2n+1} = \frac{(-1)^n a_1}{2 \cdot 4 \cdot 6 \cdot \ldots (2n)}$ for $n \geq 1$.

So the general solution is

$$\begin{aligned} y &= a_0 \left[1 - \frac{x^2}{1} + \frac{x^4}{1 \cdot 3} - \frac{x^6}{1 \cdot 3 \cdot 5} + \ldots \right] + a_1 \left[x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \ldots \right] \\ y &= a_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \ldots \cdot (2n-1)} \right] + a_1 \left[x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \ldots \cdot (2n)} \right] \end{aligned}$$