Day 58

Math 251, Spring 2005, TEST 5 **Answers**

1. Find the eigenvalues and eigenfunctions of the boundary value problem. $\lambda \geq 0$. *(20 pts)*

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$$

let $\lambda = 0$

$y'' = 0$

$y = Ax + B$

$y(0) = B = 0$

$y = Ax$

$y' = A$

$y'(\pi) = A = 0$

$y = 0$

trivial solution

let $\lambda > 0 = \mu^2$

$y'' + \mu^2 y = 0$

$y = c_1 \cos \mu x + c_2 \sin \mu x$

$y(0) = c_1 = 0$ so $c_1 = 0$

$y = c_2 \sin(\mu x)$

$y' = \mu c_2 \sin(\mu x)$

$y'(\pi) = \mu c_2 \cos(\mu \pi) = 0$

$\Rightarrow$ need $\cos(\mu\pi) = 0$

$\Rightarrow \mu\pi = \dfrac{\text{odd } \pi}{2}$

$\mu = \dfrac{2n-1}{2}$

$\mu_n = \dfrac{2n-1}{2}$

$\lambda_n = \left(\dfrac{2n-1}{2}\right)^2$

$\Rightarrow$ $\boxed{\lambda_n = \left(\frac{2n-1}{2}\right)^2 \quad y_n = \sin\left(\frac{2n-1}{2}x\right) \quad \text{for } n = 1, 2, 3, \ldots}$

2. Find the coefficients of the Fourier series for *(20 pts)*

$$f(x) = x \quad \text{for} \quad -3 \leq x < 3 \quad \text{and} \quad f(x+6) = f(x)$$

$a_0 = \dfrac{1}{3}\displaystyle\int_{-3}^{3} x\, dx = 0$

odd

$a_n = \dfrac{1}{3}\displaystyle\int_{-3}^{3} \underbrace{x \cos\left(\frac{n\pi x}{3}\right)}_{\text{odd}} dx$ (odd, even)

$a_n = 0$

$b_m = \dfrac{1}{3}\displaystyle\int_{-3}^{3} \underbrace{x \sin\left(\frac{m\pi x}{3}\right)}_{\text{even}} dx$ (odd, odd)

$b_m = \dfrac{2}{3}\displaystyle\int_0^3 x \sin\left(\frac{m\pi x}{3}\right) dx$

$u = x \qquad v = \dfrac{-3}{m\pi}\cos\left(\dfrac{m\pi x}{3}\right)$

$du = dx \qquad dv = \sin\left(\dfrac{m\pi x}{3}\right)dx$

$b_m = \dfrac{2}{3}\left[\dfrac{-3x}{m\pi}\cos\left(\dfrac{m\pi x}{3}\right)\right]_0^3 + \underbrace{\dfrac{2}{3}\dfrac{3}{m\pi}\displaystyle\int_0^3 \cos\left(\frac{m\pi x}{3}\right)dx}_{=0}$

$= \dfrac{-2}{m\pi}\left(x\cos\left(\dfrac{m\pi x}{3}\right)\right]_0^3$

$= \dfrac{-2}{m\pi}\left(3\cos(m\pi) - 0\right)$

$\boxed{= \dfrac{-6}{m\pi}\cos(m\pi) = \dfrac{6}{m\pi}(-1)^{m+1}}$ either

3. Find the solution of the heat conduction problem. *(20 pts)*

$$3u_{xx} = u_t, \quad u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad \text{and} \quad u(x,0) = 5 - 2\cos(4\pi x)$$

$3X''T = XT'$

$\dfrac{X''}{X} = \dfrac{T'}{3T} = -\lambda$

$X'' + \lambda X = 0$

$X'(0) = 0$

$X'(1) = 0$

cheat sheet

$\lambda_n = (n\pi)^2$

$X_n = \cos(n\pi x)$

$n = 0, 1, 2, \ldots$

$T' + 3\lambda T = 0$

$T = e^{-3\lambda t}$

$T_n = e^{-3\lambda_n t}$

$T_n = e^{-3(n\pi)^2 t}$

$u_n = X_n T_n$

$u_n = \cos(n\pi x) e^{-3(n\pi)^2 t}$

$u = \displaystyle\sum_{n=0}^{\infty} c_n u_n$

need c_n so that $u(x,0) =$

well

$u(x,0) = \displaystyle\sum_{n=0}^{\infty} c_n \cos(n\pi x)$

$= c_0 + c_1\cos(\pi x) + \ldots c_4 \cos(4\pi x) + \ldots =$

so $c_0 = 5$

and $c_4 = -2$

$c_n = 0$ all others.

$\boxed{u(x,t) = 5 - 2\cos(4\pi x) e^{-3(4\pi)^2 t}}$

4. The method of separation of variables can be used to replace the following differential equation with a pair of ordinary differential equations. Find this pair of equations. *(10 pts)*

let $u = XT$

then $u_{xx} + u_{xt} + u_t = 0$

$$u_{xx} + u_{xt} + u_t = 0$$

$X''T + X'T' + XT' = 0$

$X''T + (X' + X)T' = 0$

$\frac{X''}{(X'+X)} = -\frac{T'}{T} = -\lambda$

$\frac{X''}{X'+X} = -\lambda$

$X'' = -\lambda(X'+X)$

$X'' + \lambda X' + \lambda X = 0$

$-\frac{T'}{T} = -\lambda$; $-T' + \lambda T = 0$

" $T' - \lambda T = 0$

your answer: Not unique

X equation: $X'' + \lambda X' + \lambda X = 0$

T equation: $T' - \lambda T = 0$

5. Consider the following heat conduction problem with non-homogenous boundary conditions. *(10 pts)*

$$u_{xx} = u_t, \quad 0 < x < 2, \quad t > 0;$$
$$u(0,t) = 20, \quad u(2,t) = 40, \quad t > 0;$$
$$u(x,0) = 10 + 5x, \quad 0 < x < 2$$

(a) Find the steady-state solution.

$V_{xx} = 0$

$V = Ax + B$

$V(0) = B = 20$

$V = Ax + 20$

$V(2) = 2A + 20 = 40$

$A = 10$

$V(x) = 10x + 20$

your answer:

$V(x) = 10x + 20$

(b) Find the boundary value problem that determines the transient solution. This is a differential equation, boundary conditions, and an initial condition.

$u = v + w \Rightarrow w = u - v$

get $w_{xx} = w_t$

$w(0,t) = u(0,t) - v(0) = 20 - 20 = 0$

$w(2,t) = u(2,t) - v(2) = 40 - 40 = 0$

$w(x,0) = u(x,0) - v(x) = (10 + 5x) - (10x + 20)$

$= -5x - 10$

your answer:

differential equation:

$w_{xx} = w_t$

boundary conditions:

$w(0,t) = 0$

$w(2,t) = 0$

initial condition:

$w(x,0) = -5x - 10$