Math 251, Spring 2005, TEST 4 **Answers**

1. Find the Laplace transform of the following functions.

(a) $f(t) = \begin{cases} 0 & t < 2 \\ t^2 - 4t + 1 & t \geq 2 \end{cases} = u_2(t)\,[t^2 - 4t + 1] = u_2(t)\,\underbrace{[(t-2)^2 - 3]}_{f(t-2)}$ From #13

so $f(t) = t^2 - 3$

$= e^{-2s}\,\mathcal{L}\{t^2 - 3\} = \boxed{e^{-2s}\left[\frac{2}{s^3} - \frac{3}{s}\right]}$

(b) $f(t) = t^2\,\delta(t-3)$ Note: δ is the Dirac δ function we discussed in class.

but $\int_0^\infty g(t)\,\delta(t-c)\,dt = g(c)$

$\mathcal{L}\{f(t)\} = \int_0^\infty t^2\,\delta(t-3)\,e^{-st}\,dt = 3^2\, e^{-3s} = \boxed{9e^{-3s}}$ or use #19

2. Find the Laplace transform of the solution to the initial value problem given below. **Simplify**

$y'' + y = u_\pi(t)\,(t-3), \quad y(0) = y'(0) = 0$

$y'' + y = u_\pi(t)\,[\underbrace{t - \pi + (\pi - 3)}_{f(t-\pi)}]$ where $f(t) = t + \pi - 3$

$\mathcal{L}$

$[s^2 Y(s) - s\overset{0}{y(0)} - \overset{0}{y'(0)}] + Y(s) = e^{-\pi s}\left(\frac{1}{s^2} + \frac{\pi - 3}{s}\right)$

$(s^2+1)Y(s) = e^{-\pi s}\left(\frac{1}{s^2} + \frac{\pi-3}{s}\right)$

$$\boxed{Y(s) = \frac{e^{-\pi s}}{s^2+1}\left(\frac{1}{s^2} + \frac{\pi-3}{s}\right) \quad \text{or} \quad \frac{e^{-\pi s}\,(1 + (\pi-3)s)}{(s^2+1)s^2}}$$

3. Find the Laplace transform of the solution to the initial value problem given below. **Simplify**

$y'' + 2y' - y = 1, \quad y(0) = 1, \quad y'(0) = 2$

$[s^2 Y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{2}] + 2[sY(s) - \underbrace{y(0)}_{1}] - Y(s) = \frac{1}{s}$

$[s^2 + 2s - 1]\,Y(s) - s - 2 - 2 = \frac{1}{s}$

$[s^2 + 2s - 1]\,Y(s) = \frac{1}{s} + s + 4 = \frac{s^2 + 4s + 1}{s}$

$$\boxed{Y(s) = \frac{s^2 + 4s + 1}{s(s^2 + 2s - 1)}}$$

Find the inverse Laplace transform of the following functions.

(a) $F(s) = \dfrac{s+1}{s^2+4s+13} = \dfrac{s+1}{(s^2+4s+4)+13-4} = \dfrac{s+1}{(s+2)^2+9} = \dfrac{s+2}{(s+2)^2+9} - \dfrac{1}{(s+2)^2+9}$

$= \dfrac{s+2}{(s+2)^2+9} - \dfrac{1}{3}\dfrac{3}{(s+2)^2+9} \xrightarrow{\mathcal{L}^{-1}} \boxed{e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t}$

(b) $F(s) = \dfrac{2s+3}{s^2+s-6} = \dfrac{2s+3}{(s+3)(s-2)}$

$\dfrac{2s+3}{(s+3)(s-2)} = \dfrac{A}{s+3} + \dfrac{B}{s-2}$

$(2s+3) = A(s-2) + B(s+3)$

let $s=2 \to B = 7/5$

let $s=-3 \to A = 3/5$

$F(s) = \frac{3}{5}\left(\frac{1}{s+3}\right) + \frac{7}{5}\left(\frac{1}{s-2}\right)$

$\mathcal{L}^{-1}$ $\boxed{f(t) = \frac{3}{5}e^{-3t} + \frac{7}{5}e^{2t}}$

(c) $F(s) = e^{-2s}\left(\dfrac{1}{s} + 2\dfrac{1}{s^2+2}\right)$

$\downarrow \mathcal{L}^{-1}$

$f(t) = u_2(t)\, g(t-2)$

where $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} + 2\frac{1}{s^2+2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{\sqrt{2}}\frac{\sqrt{2}}{s^2+2}\right\}$

$= 1 + 2/\sqrt{2}\,\sin(\sqrt{2}t)$

$= 1 + \sqrt{2}\sin(\sqrt{2}t)$

so

$\boxed{f(t) = u_2(t)\left[1 + \sqrt{2}\sin(\sqrt{2}(t-2))\right]}$

5. Express the solution of the given initial value problem in terms of a convolution integral. You will not be able to evaluate the convolution integral.

$y'' - 4y = g(t) \qquad y(0) = y'(0) = 0$

$s^2Y(s) - sY(0) - Y'(0) - 4Y(s) = G(s)$

where $G(s) = \mathcal{L}\{g(t)\}$

$[s^2-4]Y(s) = G(s)$

$Y(s) = \frac{1}{s^2-4}G(s)$

$Y(s) = F(s)G(s)$

where $F(s) = \frac{1}{s^2-4} = \frac{1}{2}\frac{2}{s^2-4} \xrightarrow{\mathcal{L}^{-1}} f(t) = \frac{1}{2}\sinh(2t)$

by #16

$Y(t) = \frac{1}{2}\int g(t-\tau)\sinh(2\tau)\,d\tau$

or $Y(t) = \frac{1}{2}\int g(\tau)\sinh(2(t-\tau))\,d\tau$

6. Prove that if $s > 0$ then the Laplace transform of $f(t) = 1$ is $F(s) = 1/s$. Ie. Prove $\mathcal{L}\{1\} = \dfrac{1}{s}$

$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty} -\frac{1}{s}e^{-st}\Big]_0^b = -\frac{1}{s}\left[\lim_{b\to\infty} e^{-sb} - 1\right]$

now $\lim_{b\to\infty} e^{-sb} = 0$ provided $s>0$ which it is. Therefore ↑ $\mathcal{L}\{1\} = 1/s$