Math 251, TEST 1 **Name:** Answers

Graphing calculators are allowed during this exam, **except TI-89's and TI-92's**. All non-trivial answers must be justified by relevant work. In these non-trivial cases, a correct answer without any work will receive **no credit**. There is no penalty for guessing on the true-false questions.

1. Solve the following initial value problems. Find the explicit form of the solution. Ie. Express the solution as $y = f(x)$ or $y = f(t)$. *(30 pts)*

   (a) $ty' + 2y = 1, \quad y(1) = \frac{5}{2}, \quad t > 0.$

   $y' + \frac{2}{t}y = \frac{1}{t}$

   $\mu = e^{\int 2/t\, dt} = e^{2\ln t} = e^{\ln t^2} = t^2$

   $(t^2 y)' = t^2 \cdot 1/t = t$

   $t^2 y = \frac{t^2}{2} + c$

   $y = \frac{1}{2} + \frac{c}{t^2}$

   $y(1) = \frac{1}{2} + c = 5/2$

   $c = 2$

   $\boxed{y = \frac{1}{2} + \frac{2}{t^2}}$

   (b) $y' = \dfrac{2x}{y(1+x^2)}, \quad y(0) = -2$ (separable)

   $y\,dy = \frac{2x}{1+x^2}\,dx$

   $\int\int \quad \frac{y^2}{2} = \ln(1+x^2) + c$

   $\Rightarrow y = \pm\sqrt{2\ln(1+x^2) + c}$

   $y(0) = \pm\sqrt{c} = -2$ choose $\ominus$ and $c = 4$

   $\boxed{y = -\sqrt{2\ln(1+x^2) + 4}}$

2. Consider the differential equation *(15 pts)*

$$\overset{M}{(xy^2 + bxy)} + \overset{N}{(x^2 + x^2y + 1)}\frac{dy}{dx} = 0$$

   (a) Find the value of b for which the differential equation is exact.

   $M_y = 2xy + bx \qquad N_x = 2x + 2xy \Rightarrow b = 2$

   (b) Solve the differential equation for this specific value of b.

   $\psi_x = M = xy^2 + 2xy$

   $\psi = \frac{x^2y^2}{2} + x^2y + h(y)$

   $\psi_y = x^2y + x^2 + h'(y)$

   but $\psi_y = N = x^2 + x^2y + 1$

   $\Rightarrow h'(y) = 1$

   $h(y) = y$

   solution is $\psi(x,y) = c$

   $\boxed{\frac{x^2y^2}{2} + x^2y + y = C}$

3. Determine (without solving the initial value problem) the largest interval in which the solution to the given initial value problem is **certain** to exist. *(5 pts)*

   $$(t-3)y' + \ln(t)\,y = 2t, \quad y(1) = 2$$

   $y' + \frac{\ln(t)}{t-3}y = \frac{2t}{t-3}$

   need $t > 0$ because $\ln(t)$ and $t \neq 3$

   The interval must contain 1

   $\boxed{(0,3)}$

4. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Express Newton's law of cooling as a **differential equation**. Let t denote time, $T(t)$ denote the temperature of the object, and T_A denote the temperature of its surroundings. Be sure to designate the sign of any constants. *(5 pts)*

   $dT/dt = -K(T - T_A) \qquad K > 0$

5. A tank with a capacity of 1500 gal originally contains 100 gal of water with 50 lb of salt in solution. Water containing γ lbs of salt per gallon is entering at a rate of 5 gal/min and the well mixed solution is allowed to flow out of the tank at 3 gal/min.

Let t denote time prior to the tank overflowing and $Q(t)$ denote the pounds of salt in the tank during that time. Derive the **initial value problem** with $Q(t)$ as the solution. *(5 pts)*

$Q' =$ rate in $-$ rate out $\quad V' = 5-3 = 2 \quad V = 100 + 2t$

$= 5\gamma - 3 \cdot Q/V \quad V = 2t + c$

$\boxed{Q' = 5\gamma - 3Q/(100+2t) \quad Q(0) = 50}$

6. Circle **T** if the statement is true and **F** if the statement is false.

AMBIGUOUS MARKS $\Longleftrightarrow$ WRONG ANSWER *(6 pts)*

(a) **T** **F** $y(t) = \cos 3t + 1$ is a solution to the differential equation $y'' + 9y = 0$. Check it

(b) **T** **F** $y(t) = e^{2t}$ is a solution to the differential equation $y'' - y' - 2y = 0$. Check it

(c) **T** **F** Suppose $y = \phi(t)$ is a solution to the differential equation $y' = f(t,y)$. If f and $\partial f/\partial y$ are continuous for all t then ϕ is continuous for all t. Thm 2.4.2

7. Consider the autonomous differential equation $y' = f(y)$ defined by *(14 pts)*

$$\frac{dy}{dt} = (e^y - 1)\,(y-3).$$

(a) Make a rough sketch of the graph of $f(y)$ versus y.

(b) Determine the equilibrium solution(s) and classify each one as unstable or asymptotically stable. Equilibrium solutions are sometimes called critical points or constant solutions.

$y = 0$ asymptotically stable

$y = 3$ unstable

(c) Sketch several graphs of solutions in the ty-plane.