

1. Both of these problems deal with the differential equation:

(20 pts)

$$y^{(4)} - 3y'' - 4y = g(t).$$

(a) Find the general solution to $y^{(4)} - 3y'' - 4y = 0$

assume $y = e^{rt}$ get

$$r^4 - 3r^2 - 4 = 0$$

$$(r^2 - 4)(r^2 + 1)$$

$$\left. \begin{matrix} r = 2 \\ r = -2 \end{matrix} \right\} e^{2t} + e^{-2t}$$

$$\left. \begin{matrix} r = i \\ r = -i \end{matrix} \right\} \cos t + \sin t$$

$$Y = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos t + c_4 \sin t$$

(b) Find a suitable form for a particular solution $Y(t)$ if the method of undetermined coefficients is to be used in solving $y^{(4)} - 3y'' - 4y = 3t + e^{2t}$.

$Y_1 = At + B$ corresponding to $3t$

$Y_2 = Ct e^{2t}$ corresponding to e^{2t}

$$Y = At + B + Ct e^{2t}$$

2. Consider the differential equation

(20 pts)

$$(2+x^2)y'' + 4y = 0$$

(a) Use the Power Series Method to derive the recursive relationship between coefficients in the power series representation of the solution.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2}$$

$$(2+x^2) \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2} + 4 \sum_{n=0}^{\infty} a_n x^n$$

$$2 \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2} + x^2 \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2} + \sum_{n=0}^{\infty} 4a_n x^n$$

$$\sum_{n=0}^{\infty} 2(n+1)(n+2)a_{n+2} x^n + \sum_{n=2}^{\infty} (n-1)n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n$$

↓ * trick

$$\sum_{n=0}^{\infty} (n-1)n a_n x^n$$

plug into D.E.

get $2(n+1)(n+2)a_{n+2} + (n-1)n a_n + 4a_n = 0$

(b) Use either the Power Series Method or the Taylor Series Method to find the first 4 nonzero terms in the power series for the solution satisfying the initial conditions

Power Series $a_0 = y(0) = 1$
 $a_1 = y'(0) = 2$

and

$$a_{n+2} = -\frac{(n-1)n + 4}{2(n+1)(n+2)} a_n$$

$$a_2 = -\frac{4}{4} a_0 = -1$$

$$a_3 = -\frac{4}{12} a_1 = -\frac{2}{3}$$

$$y = 1 + 2x - x^2 - \frac{2}{3}x^3 + \dots$$

$y(0) = 1$ and $y'(0) = 2$

Taylor Series

$$y'' = -4y/(2+x^2)$$

$$y''(0) = -4y(0)/2 = -2$$

$$y''' = \frac{(2+x^2)(-4y') + 4y(2x)}{(2+x^2)^2}$$

$$y'''(0) = -16/4 = -4$$

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

$$y = 1 + 2x - x^2 - \frac{2}{3}x^3 + \dots$$

3. Consider the differential equation

$$\overbrace{x^2(1-x)y'' + (x-2)y' - 3xy = 0}^{\text{set } = 0}$$

(15 pts)

- (a) What are the singular points of this differential equation. $x=0, x=1$
 (b) Classify each as regular or irregular. Show work justifying each answer.

$\Rightarrow \lim_{x \rightarrow 0} \frac{x(x-2)}{x^2(1-x)} = \lim_{x \rightarrow 0} \frac{x-2}{x(1-x)}$ Not finite $x=0$ is irregular

$\Rightarrow \lim_{x \rightarrow 1} \frac{(x-1)(x-2)}{x^2(1-x)} = -\frac{(x-2)}{x^2} = 1 \checkmark$
 $\lim_{x \rightarrow 1} \frac{(x-1)^2(x-2)}{x^2(1-x)} = -\frac{(x-1)(x-2)}{x^2} = 0 \checkmark$ $x=1$ is regular

4. Find the general solution to the differential equation: $x^2y'' + 4xy' + 2y = 0, x > 0$. (5 pts)

$y = x^r$
 $y' = rx^{r-1}$
 $y'' = r(r-1)x^{r-2}$

$$x^2[r(r-1)x^{r-2}] + 4rx^{r-1} + 2x^r = 0$$

$$r(r-1)x^r + 4rx^r + 2x^r = 0$$

$$(r(r-1) + 4r + 2)x^r = 0$$

$$r^2 - r + 4r + 2 = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$$r = -2, -1$$

$y = c_1x^{-2} + c_2x^{-1}$ (15 pts)

5. Consider the differential equation defined by

$$xy'' + \alpha y' = 0 \quad x > 0 \quad \text{and} \quad \alpha \neq 1$$

This is an Euler Equation. Solutions can be found by assuming a solution of the form $y = x^r$. When this is done you get a general solution defined by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where $y_1(x) = 1$.

(a) Find $y_2(x)$. Your answer will contain α .

$y = c_1x^0 + c_2x^{1-\alpha}$
 $\frac{1}{y_1}$
 so $y_2 = x^{1-\alpha}$

$$xy'' + \alpha y' = 0$$

$$x(r-1)r x^{r-2} + \alpha r x^{r-1} = 0$$

$$(r-1)r x^{r-1} + \alpha r x^{r-1} = 0$$

$$(r-1)r + \alpha r = 0$$

$$r(r-1+\alpha) = 0$$

$$r = 0 \quad \text{or} \quad r = 1-\alpha$$

(b) Determine the values of α which guarantee that $\lim_{x \rightarrow 0} y_2(x) = 0$.

$\lim_{x \rightarrow 0} x^{1-\alpha} = 0$ provided $1-\alpha > 0$ or $\alpha < 1$

6. Find the radius of convergence (ρ) of the power series $\sum_{n=0}^{\infty} \frac{(3x-1)^n}{n^2}$. (5 pts)

$\lim_{n \rightarrow \infty} \left| \frac{(3x-1)^{n+1}}{(3x-1)^n} \cdot \frac{n^2}{(n+1)^2} \right| = |3x-1| \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = |3x-1|$
 need $r < 1$ so $|3x-1| < 1 \rightarrow |x-1/3| < 1/3$
 $\rho = 1/3$