Math 251, Spring 2005, TEST 2 **Answers:**

1. Consider the differential equation (10 pts)

$$y'' + p(t)\,y' + q(t)\,y = g(t),$$

the initial conditions

$$y(t_o) = y_o \quad \text{and} \quad y'(t_o) = y'_o,$$

and an open interval

$$I : \alpha < t < \beta$$

which contains t_o, on which p, q, and g are continuous.

(a) **F** There are infinitely many solutions to the differential equation which satisfy the initial conditions. False, there are infinitely many solutions to the diff eq but a unique solution to the initial value problem.

(b) **F** If $y_o = y'_o = 0$, then the only solution to the initial value problem is $\phi(t) = 0$. False, 0 is not even a solution unless $g(t) = 0$.

(c) **F** If $g(t) = 0$, and y_1 and y_2 are both solutions of the differential equation then so is $\phi(t) = y_1(1 + y_2)$. This is a False Statement because $y_1 \cdot y_2$ is not necessarily a solution.

(d) **T** If $g(t) = 0$, and y_1 and y_2 are both solutions of the differential equation such that $W[y_1, y_2] = 1$ at $t = t_o$, then y_1 and y_2 are linearly independent on I. True.

(e) **F** If $g(t) = t^2$ and y_1 and y_2 are solutions to the differential equation, then $\phi(t) = y_1 + y_2$ is also a solution. False. If $g(t) = 0$, then the implication would hold true but that is not the case here.

2. For each differential equation, circle **all** solutions. (15 pts)

(a) $y'' + 2y' + y = 0$: Note $y_1 = e^{-t}$ and $y_2 = te^{-t}$.
$\phi_1(t) = e^{-t}$ $\quad \phi_2(t) = e^{2t}$ $\quad \phi_3(t) = te^{2t}$ $\quad \phi_4(t) = (3t + 4)e^{-t}$ $\quad \phi_5(t) = 0$

(b) $y'' + 4y' = 0$: Note $y_1 = 1$ and $y_2 = e^{-4t}$.
$\phi_1(t) = 3e^{-t}$ $\quad \phi_2(t) = 7$ $\quad \phi_3(t) = \cos(2t)$ $\quad \phi_4(t) = 2 + 4e^{-4t}$ $\quad \phi_5(t) = te^{-4t}$

(c) $y'' - 4y' + 5y = 0$ Note: $r^2 - 4r + 5 = 0$ for $r = 2 \pm i \rightarrow y_1 = e^{2t}\cos t$ and $y_2 = e^{2t}\sin t$.
$\phi_1(t) = e^t \cos 2t$ $\quad \phi_2(t) = e^{2t}\cos t$ $\quad \phi_3(t) = e^{2t}\cos 2t$ $\quad \phi_4(t) = e^t \cos t$ $\quad \phi_5(t) = e^{4t}$

3. Find a second linearly independent solution by "reduction of order" (20 pts)

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0, \quad y_1(t) = t^{-1}$$

$y_2 = vt^{-1}$

$y_2' = v't^{-1} - vt^{-2}$

$y_2'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$

$2t^2[v''t^{-1} - 2v't^{-2} + 2vt^{-3}] + 3t[v't^{-1} - vt^{-2}] - vt^{-1} = 0$

$2v''t - 4v' + 4vt^{-1} + 3v' - 3vt^{-1} - vt^{-1} = 0$

$2t\,v'' - v' = 0 \Rightarrow u' - \frac{1}{2t}u = 0 \Rightarrow du/u = \frac{1}{2t} \rightarrow \ln|u| = \ln t^{1/2} + c$

$u = c_1 t^{1/2}$ where $u = v' \Rightarrow v = \frac{2}{3}c_1 t^{3/2} + c_2$

let $v = t^{3/2}$ Then $y_2 = t^{3/2} \cdot t^{-1} = t^{1/2}$

$y_2 = t^{1/2}$

4. Determine the **form** of the particular solution, **Y**. *(20 pts)*

(a) $y'' + 5y' + 6y = 3t^2e^{-7t}$ $\rightarrow y_1 = e^{-2t}, y_2 = e^{-3t}$ $\rightarrow$ $Y = (A_0t^2 + A_1t + A_2)e^{-7t}$

(b) $y'' + 5y' + 6y = e^{-3t}$ " " $\rightarrow$ $Y = Ate^{-3t}$

(c) $y'' + 4y' + 4y = (3t+1)e^{-2t}$ $y_1 = e^{-2t}, y_2 = te^{-2t}$ $Y = t^2(A_0t + A_1)e^{-2t}$

(d) $y'' + 4y = \cos(3t)$ $y_1 = \cos 2t$ $y_2 = \sin 2t$ $Y = A\cos 3t + B\sin 3t$

5. Use the *method of undetermined coefficients* to find a particular solution **Y** of the given nonhomogeneous equation. *(10 pts)*

$r^2 + 2r - 3 = (r+3)(r-1)$

$y_1 = e^{-3t}$ $y_2 = e^{t}$

$$y'' + 2y' - 3y = 10e^{2t}.$$

Assume $Y = Ae^{2t}$

$Y' = 2Ae^{2t}$

$Y'' = 4Ae^{2t}$

$4Ae^{2t} + 4Ae^{2t} - 3Ae^{2t} = 10e^{2t}$

$5A = 10$

$A = 2$

$\rightarrow$ $\boxed{Y = 2e^{2t}}$

6. Consider the differential equation *(15 pts)*

$$y'' - 4y = 0$$

(a) Find the *general solution* to the differential equation.

$r^2 - 4 = 0$ $y_1 = e^{2t}$

$r = \pm 2$ $y_2 = e^{-2t}$

$\boxed{Y = C_1e^{2t} + C_2e^{-2t}}$

(b) Find the solution which satisfies the initial conditions

$$y(0) = \alpha, \quad y'(0) = 2$$

$Y' = 2C_1e^{2t} - 2C_2e^{-2t}$

$\left.\begin{array}{l} Y(0) = C_1 + C_2 = \alpha \\ Y'(0) = 2C_1 - 2C_2 = 2 \end{array}\right\}$

① $C_1 + C_2 = \alpha$

② $C_1 - C_2 = 1$

① + ② $\rightarrow$ $2C_1 = \alpha + 1$

$\rightarrow C_1 = \frac{\alpha+1}{2}$

$C_2 = C_1 - 1 = \frac{\alpha+1}{2} - 1$

$\rightarrow C_2 = \frac{\alpha-1}{2}$

$\boxed{Y = \frac{\alpha+1}{2}e^{2t} + \frac{\alpha-1}{2}e^{-2t}}$

(c) Find α so that the solution tends to zero as $t \rightarrow \infty$.

In order that $y \rightarrow 0$ as $t \rightarrow \infty$ you need the coefficient of e^{2t} to be zero.

Ie Need $\frac{\alpha+1}{2} = 0$ $\rightarrow$ $\boxed{\alpha = -1}$