Final Assignment

MTHBD/CMPBD 423

Due: Thursday December 15 at 10:00 AM

1. The sensitivity of the Hilbert matrix. *(20 pts)*

   The Hilbert matrix is a notoriously ill-conditioned matrix. It arises in finding a polynomial of best fit in the least squares sense. It is square and symmetric. It's elements are defined by $H_{i,j} = \frac{1}{i+j-1}$. The command hilb(n) will generate this matrix. Associated with each matrix you will create an n-vector $b = Hx$, where x is the n-vector with all ones. This vector can be created with the command x = ones(n,1). The goal is to solve $H\tilde{x} = b$ for $\tilde{x}$ using MATLAB linear solving command H $\backslash b$. Keep in mind, we know the exact solution and it is the vector of all ones = x. Fill in the table on the cover page and answer the two questions afterwards. This is designed to show you how the condition number gets very large with large n and this results in large residuals and errors. The condition number is to be given with respect to the infinity norm: k = cond(H,inf). The residual and errors are to be calculated using the infinity norm as well: norm(vector,inf).

   Note: You should get a warning from MATLAB when $n \geq 12$ stating that the matrix is either close to singular or badly scaled. It is, as noted above, ill-conditioned.

   Hand in the completed table on the next page with answers to the two question. Do not hand in any code.

2. Approximate *(20 pts)*

$$\int_0^{\pi/2} \sin(x)\, dx$$

   using the Trapezoid rule, Simpson's 1/3 rule, and Gaussian quadrature with two nodes. You are not to use the composite formulas, just one step with each. Give the approximations and error for each method in the table on the cover page. Which is the best method and why?

3. Prove that the condition number of an invertible must be at least one. *(10 pts)*

4. Find a 3x3 invertible matrix A such that $1 \leq k(A) \leq 2$ and $\det(A) \leq \frac{1}{10}$ *(10 pts)*

5. Suppose you have a matrix A which is $n\ x\ n$ and an n-vector b. Also assume that you are given A^{-1} as well as the LU-factors of $A = LU$. *(20 pts)*

   (a) Show that the number of flops in calculating $A^{-1}b$ is $2n^2 - n$.

   (b) Now calculate the number of flops needed solve $LUx = b$ for x by first solving $Ly = b$ for y and then $Ux = y$ for x. Justify your answer.

   (c) Anything interesting about these results?

6. Recall that the definition of a matrix norm is based on that of a vector norm by *(20 pts)*

$$||A|| = \text{Sup}\{\ ||Ax||\ :\ ||x|| = 1\ \}.$$

Use this definition and the definition of the $||x||_1$ to prove that the ℓ_1 norm of an n x n matrix is defined by

$$||A||_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |A_{ij}|.$$

Hint: First shown that $||A||_1 \le \max_{1\le j\le n} \sum_{i=1}^{n} |A_{ij}|$. Then show that equality holds for a certain x vector of norm one.